On persistency of excitation for uncontrollable systems and networks

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***Abstract*— Recent work on data-driven methods based on the work of Willems *et al.* has extended the so-called *fundamental lemma* to include uncontrollable systems. We prove a lower requirement on persistency of excitation of the input signal for certain uncontrollable systems, replacing the order of the system by the degree of controllability. Additionally, we highlight the usefulness of these results to data-driven control of large-scale networks through the notion of structural controllability.**

I. Introduction

The use of data to identify and control linear systems is prevalent in the field of control theory. For example, much work has been done to accommodate a more data-centric approach to these topics within the field of adaptive control [1]. An alternative to these model-based methods was pioneered by Willems et al. [2]. In their work, the authors show that the dynamics of a system can be fully characterized by a single trajectory of sufficient length, given a sufficiently exciting input signal. In distancing itself from the more traditional methods, this *behavioral approach* seeks to remove the need to explicitly model a system and instead establish an entirely data-driven framework.

Recent progress has been made in several areas extending both the theoretical basis and application of data-driven methods based on the work of Willems et al., see [3] for an overview. Interest in the behavioral approach to control and identification was renewed by works showing its applicability to existing methods, for example [4] and [5] presenting new methods for data-driven counterparts to MPC. In [6], the authors present results that allow for the use of multiple trajectories of different lengths to replace the impractical single trajectory demanded by the original formulation. The recent work [7] extends the fundamental lemma to apply to uncontrollable systems, which was not possible in the previous framework. Additionally, the work in [7] contains reduced requirements on the persistency of excitation of the input signal for systems with a specific structure. This work also focuses on such requirements for uncontrollable systems.

The role of controllability and its effect on data-driven methods has recently been explored, for example in [8]. Therein, the authors propose an algorithm for determining the controllability of a system. While loss of controllablility is frequently seen as a flaw in the design of a system, for many applications it is an inherent feature. In large-scale network systems, where the available actuation might be very sparse or local, uncontrollability to some degree may be unavoidable and even desirable. The notion of *structural controllability* is proposed in [9] and further developed in [10] to analyze the controllability of large networks without detailed knowledge of the system parameters. The well-received work in [11] details the empirical study of several real large-scale networks using structural controllability and finds that many of these require actuation in a large fraction of the nodes in order to achieve controllability. As a consequence, a loss of controllability is guaranteed and quantifiable if actuation is only available in a small subset of the nodes. In the realistic setting, a system may lose further degrees of controllability from modes that are theoretically controllable but difficult to excite in practice.

The work presented in this paper is an intuitive addition to the results from [7], further reducing the requirement on persistency of excitation for uncontrollable systems. This is achieved by accounting for the fact that the dynamics of an uncontrollable system, under certain assumptions on the initial condition, will take place in a subspace of the state space. The input-output relation of this lower-dimensional behavior is possible to characterize with less data than a controllable system would require. We also demonstrate this fact through a numerical example.

The paper begins by describing the problem to be analyzed in Section II together with definitions of necessary tools and constructions. The main contribution of this paper in extending previous results, together with the required assumptions, is detailed in Section III. A numerical example is constructed in Section IV to demonstrate the validity of the presented results. In Section V, the applicability of the results to real network systems is discussed and the presented results are compared to previous results. Finally, conclusions and potential directions for future work are presented in Section VI.

A. Notation

Consider a matrix $M \in \mathbb{R}^{p\times q}$. Its image, kernel and rank are denoted $\text{im}(M)$, $\ker(M)$ and $\text{rank}(M)$ respectively. The entry at the i-th row and j-th column of M is denoted $m_{i,j}$. Let $\lambda_i(M)$ denote the i-th eigenvalue of M. When it is clear from context which matrix is referred to we will write simply λ_i. Given a discrete signal $x_t \in \mathbb{R}^p$ we write $x_{[t_1,t_2]} = \begin{bmatrix} x_{t_1}, x_{t_1+1}, ..., x_{t_2}\end{bmatrix}$ to refer to the part of the signal between times t_1 and t_2. We denote the matrix of size $p \times q$ of all zeros as $0_{p\times q}$, the vector of size $p\times 1$ of all zeros as 0_p and omit the size when it is evident.

II. Preliminaries

The problem concerns the identification and control of discrete-time LTI systems (with a particular interest in their interpretation as networked systems) on the form

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t \\ y_t &= Cx_t + Du_t \end{aligned} \tag{1}$$

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where $u_t \in \mathbb{R}^m$ is the control signal, $x_t \in \mathbb{R}^n$ is the state and $y_t \in \mathbb{R}^p$ is the measured output.

A. Controllability and observability

Definition 1 (Controllability matrix): The controllability matrix of the system (1) is defined as

$$\mathbf{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}. \quad (2)$$

The controllable subspace of the system is then given by $\mathcal{C} = \text{im}(\mathbf{C})$ with dimension $\dim(\mathcal{C}) = \text{rank}(\mathbf{C})$. This subspace consists of all points from which, given infinite time and control effort, it is possible to control the state to the origin. The closely associated reachable subspace, which may be more useful in explaining the intuition of the results presented here, instead consists of all points that can be reached from the origin. If A is invertible these subspaces are identical, otherwise the reachable subspace is a subset of the controllable subspace. Since the controllable subspace is, as the name suggests, a subspace of $\mathbb{R}^n$ there exists a basis of $\dim(\mathcal{C})$ linearly independent vectors spanning $\mathcal{C}$. These vectors correspond to the controllable modes of the system. The remaining $n - \dim(\mathcal{C})$ basis vectors in $\mathbb{R}^n$ constitute the kernel of $\mathbf{C}$ and span the uncontrollable subspace denoted $\overline{\mathcal{C}}$.

Observability can in a similar fashion be analyzed using the analogous observability matrix:

Definition 2 (Observability matrix): The observability matrix of the system (1) is defined as

$$\mathbf{O} = \begin{bmatrix} C^\top & (AC)^\top & \cdots & (A^{n-1}C)^\top \end{bmatrix}^\top. \quad (3)$$

The observable subspace is denoted $\mathcal{O} = \text{im}(\mathbf{O})$. Of special interest here is its complement, the unobservable subspace $\overline{\mathcal{O}} = \ker(\mathbf{O})$, which consists of all states that give a zero output of the system (1).

B. Behavioral theory

The behavioral approach to systems theory seeks to provide a model-free characterization of a dynamical system. In [2], Willems et al. formulate the result referred to as the *fundamental lemma*, which allows the parameterization of all possible future behaviors of a system by using a recorded trajectory, provided that that it is generated by an input signal that is *persistently exciting* to a sufficient degree.

Definition 3 (Hankel matrix): The depth-d Hankel matrix of a length-K signal $x_{[0,K-1]}$ is defined as

$$\mathcal{H}_d(x_{[0,K-1]}) = \begin{bmatrix} x_0 & x_1 & \cdots & x_{K-d} \\ x_1 & x_2 & \cdots & x_{K-d+1} \\ \vdots & \vdots & & \vdots \\ x_{d-1} & x_d & \cdots & x_{K-1} \end{bmatrix}$$

where $K \geq d$. Building on this definition, we can define the notion of persistent excitation.

Definition 4 (Persistency of excitation): A discrete signal $u_{[0,K-1]}$ is said to be persistently exciting of order d if the Hankel matrix $\mathcal{H}_d(u_{[0,K-1]})$ has full row rank.

Due to the structure of this matrix, a necessary condition is for the signal to be $K \geq (m+1)d - 1$ samples long, in order for $\mathcal{H}_d(u_{[0,K-1]})$ to have a sufficient number of columns to allow for the possibility of d independent row vectors.

The main application of the fundamental lemma presented in [2] is the construction of new trajectories from previously collected data. Below, we give a definition of this property.

Definition 5 (Parameterization): The input-output trajectory $(\bar{u}_{[0,T-1]}, \bar{y}_{[0,T-1]})$ can be parameterized by the trajectory $(u_{[0,T-1]}, y_{[0,T-1]})$ if there exists a vector $g \in \mathbb{R}^{T-L+1}$ fulfilling

$$\begin{bmatrix} \bar{u}_{[0,T-1]} \\ \bar{y}_{[0,T-1]} \end{bmatrix} = \begin{bmatrix} \mathcal{H}_L(u_{[0,T-L]}) \\ \mathcal{H}_L(y_{[0,T-1]}) \end{bmatrix} g. \quad (4)$$

In order to facilitate our analysis of the system (1) we now state a reformulation of the results in [2] for the state-space setting, presented in [6].

Theorem 2.1 ([6, Theorem 1]): Consider the system (1) and assume that the pair (A, B) is controllable. Let $(u_{[0,T-1]}, x_{[0,T-1]}, y_{[0,T-1]})$ be an input-state-output trajectory of (1). Assume that the input $u_{[0,T-1]}$ is persistently exciting of order $n + L$. Then the following statements hold:

(i) The matrix $\begin{bmatrix} \mathcal{H}_1(x_{[0,T-L]}) \\ \mathcal{H}_L(u_{[0,T-1]}) \end{bmatrix}$ has full row rank.

(ii) Every length L input-output trajectory of (1) can be parameterized by $(u_{[0,T-1]}, y_{[0,T-1]})$.

Note especially the requirements for the pair (A, B) to be controllable and for the signal $u_{[0,T-1]}$ to be persistently exciting of order $n + L$. The controllability condition is relaxed in [7]. In the following section, we seek to reduce the required order of persistency of excitation for systems lacking full controllability, reducing the required amount of data for data-driven methods to succeed.

III. UNCONTROLLABILITY AND PERSISTENCY OF EXCITATION

The key idea exploited in the following result is that the space in which the dynamics of an uncontrollable system unfold is a subspace of the full state space. As a consequence there are fewer modes that need to be excited by the input in order to fully characterize the relevant input-output relation. In order to present our version of Theorem 2.1 we first make an assumption on the initial condition of the system (1).

Assumption 1: Let $x_0 \in \mathcal{C}$.

This assumption is natural for many systems, since it could mean that the trajectory $(u_{[0,T-1]}, x_{[0,T-1]}, y_{[0,T-1]})$ originated from some stable setpoint of the system. Note that x_0 need not be the setpoint; it is enough to know that the system previously (for $t < 0$) originated in the stable state and was controlled from there using the same matrix B. In the setting of multiple trajectories considered in [6] this assumption can still be reasonable, for example in the application discussed therein of missing data samples. As long as we know that the trajectory originated in $\mathcal{C}$, the points after the missing samples, which are treated as the starting points of new trajectories, will also be contained in $\mathcal{C}$.

We state our main result, which gives relaxed conditions on persistency of excitation compared to Theorem 2.1 for uncontrollable systems in the following theorem:

Theorem 3.1: Consider an input-state-output trajectory $(u_{[0,T-1]}, x_{[0,T-1]}, y_{[0,T-1]})$ of a system on the form (1) and let Assumption 1 hold. If the input $u_{[0,T-1]}$ is persistently exciting of order $\dim(\mathcal{C}) + L$ then

$$\operatorname{im}\begin{bmatrix}\mathcal{H}_1(x_{[0,T-L]})\\ \mathcal{H}_L(u_{[0,T-1]})\end{bmatrix} = \mathcal{C}\times\mathbb{R}^{mL}, \tag{5}$$

and the input-output component of any trajectory $(\bar{u}_{[0,L-1]}, \bar{x}_{[0,L-1]}, \bar{y}_{[0,L-1]})$ that is a solution of (1) with

$$\bar{x}_0 \in \mathcal{C} + \overline{\mathcal{O}} \tag{6}$$

can be parameterized by $(u_{[0,L-1]}, y_{[0,L-1]})$.

Proof: Taking inspiration from [7], we use a double inclusion argument to prove (5). Starting with the left-hand side, $\operatorname{im}(\mathcal{H}_L(u_{[0,T-1]})) \subseteq \mathbb{R}^{mL}$ by definition. Under Assumption 1 we can also guarantee that $\operatorname{im}(\mathcal{H}_1(x_{[0,T-1]})) \subseteq \mathcal{C}$.

For the opposite direction we fist notice that, since the system (1) has $\dim(\mathcal{C}) \le n$ degrees of controllability, we can find some similarity transformation T such that the new basis aligns with the modes of the system. Applying this transformation we can partition the new state vector $\hat{x} = Tx$ in such a way that the first $\dim(\mathcal{C})$ states are controllable and the remaining $n - \dim(\mathcal{C})$ uncontrollable. Since clearly, $\operatorname{im}(\mathcal{H}_1(x_{[0,T-L]})) = \operatorname{im}(\mathcal{H}_1(\hat{x}_{[0,T-L]}))$, will now prove the equivalent inclusion

$$\mathcal{C}\times\mathbb{R}^{mL} \subseteq \operatorname{im}\begin{bmatrix}\mathcal{H}_1(\hat{x}_{[0,T-L]})\\ \mathcal{H}_L(u_{[0,T-1]})\end{bmatrix}. \tag{7}$$

For this purpose, let $\xi \in \mathbb{R}^n$, $\eta_i \in \mathbb{R}^m$ for $i = 1, 2, ..., L$ and split the first vector as $\xi^\top = \begin{bmatrix}\xi_{\mathcal{C}}^\top & \xi_{\overline{\mathcal{C}}}^\top\end{bmatrix}$, dividing it into elements corresponding to the controllable and uncontrollable states of $\hat{x}$. Then define a vector

$$v^\top = \begin{bmatrix}\xi_{\mathcal{C}}^\top & \xi_{\overline{\mathcal{C}}}^\top & \eta_1^\top & \eta_2^\top & \cdots & \eta_L^\top\end{bmatrix}$$

and suppose v is such that

$$v^\top\begin{bmatrix}\mathcal{H}_1(\hat{x}_{[0,T-L]})\\ \mathcal{H}_L(u_{[0,T-1]})\end{bmatrix} = 0. \tag{8}$$

The inclusion (7) can now be proven by showing that all possible vectors $v^\top$ satisfying (8) are orthogonal to $\mathcal{C}\times\mathbb{R}^{mL}$. To show this, we proceed to construct additional vectors $w_0, w_1, ..., w_{\dim(\mathcal{C})} \in \mathbb{R}^{n+m(\dim(\mathcal{C})+L)}$ and let $w_0 = \begin{bmatrix}v^\top & 0_{m\dim(\mathcal{C})}^\top\end{bmatrix}^\top$. Similar to the partition of ξ, the transformed system matrix $\hat{A} = T^{-1}AT$ is split into

$$\hat{A} = \begin{bmatrix}\hat{A}_{\mathcal{C}\mathcal{C}} & \hat{A}_{\overline{\mathcal{C}}\mathcal{C}}\\ \hat{A}_{\mathcal{C}\overline{\mathcal{C}}} & \hat{A}_{\overline{\mathcal{C}}\overline{\mathcal{C}}}\end{bmatrix} \tag{9}$$

where $\hat{A}_{\mathcal{C}\mathcal{C}} \in \mathbb{R}^{\dim(\mathcal{C})\times\dim(\mathcal{C})}$ and the sizes of the other submatrices follow. Observe that, since the transformation T divides x into states corresponding to controllable and uncontrollable modes, the controllable components in $\hat{x}$ will have no influence on the uncontrollable ones. This implies that $\hat{A}_{\mathcal{C}\overline{\mathcal{C}}} = 0$. Let the remaining vectors w_i for $i = 1, 2, ..., \dim(\mathcal{C})$ be on the form

$$\begin{bmatrix}\xi^\top\hat{A}^i & \xi^\top\hat{A}^{i-1}B & \cdots & \xi^\top B & \eta_1^\top & \cdots & \eta_L^\top & 0_{m(\dim(\mathcal{C})-i)}^\top\end{bmatrix}^\top.$$

As a consequence of equation (8) combined with repeated application of the dynamics (1) we have that all w_i fulfill

$$w_i^\top\begin{bmatrix}\mathcal{H}_1(\hat{x}_{[0,T-1]})\\ \mathcal{H}_{\dim(\mathcal{C})+L}(u_{[0,T-1]})\end{bmatrix} = 0. \tag{10}$$

Next, we make an observation regarding the matrix $\hat{A}$. The partition (9) together with the fact that $\hat{A}_{\mathcal{C}\overline{\mathcal{C}}} = 0$ means that the powers $\xi^\top\hat{A}^i$ must be on the form

$$\xi^\top\hat{A}^i = \begin{bmatrix}\xi_C^\top & \xi_U^\top\end{bmatrix}\begin{bmatrix}\hat{A}_{\mathcal{C}\mathcal{C}}^i & \star\\ 0 & \hat{A}_{\overline{\mathcal{C}}\overline{\mathcal{C}}}^i\end{bmatrix} = \begin{bmatrix}\xi_C^\top\hat{A}_{\mathcal{C}\mathcal{C}}^i & \star\end{bmatrix}. \tag{11}$$

We therefore examine a linear combination of these vectors

$$\sum_{i=0}^{\dim(\mathcal{C})}\alpha_i w_i^\top = \begin{bmatrix}\sum_{i=0}^{\dim(\mathcal{C})}\alpha_i\xi_C^\top\hat{A}_{\mathcal{C}\mathcal{C}}^i & q^\top & r^\top\end{bmatrix} \tag{12}$$

where $q \in \mathbb{R}^{n-\dim(\mathcal{C})}$ and $r \in \mathbb{R}^{m(\dim(\mathcal{C})+L)}$ collect the sums of the rightmost terms in (11) and the sum of the final $m(\dim(\mathcal{C}) + L))$ terms of the vectors w_i respectively. Here we recognize that an appropriate choice of constants α_i will, according to the Cayley-Hamilton theorem, render $\sum_{i=0}^{\dim(\mathcal{C})}\alpha_i\xi_C^\top\hat{A}_{\mathcal{C}\mathcal{C}}^i = 0_{\dim(\mathcal{C})}^\top$, since $\hat{A}_{\mathcal{C}\mathcal{C}}$ is of size $\dim(\mathcal{C})$. This, together with the fact that $q^\top$ will be multiplied only by the uncontrollable states in equation (10) (since $\hat{x} = \begin{bmatrix}\hat{x}_{\mathcal{C}}^\top & \hat{x}_{\overline{\mathcal{C}}}^\top\end{bmatrix}^\top$) that as a consequence of Assumption 1 are zero, implies that $r^\top$ must be in the left kernel of $\mathcal{H}_{\dim(\mathcal{C})+L}(u_{[0,T-1]})$. This matrix is of full row rank since we require $u_{[0,T-1]}$ to be persistently exciting of order $\dim(\mathcal{C}) + L$, which in turn implies that

$$r = 0_{m(\dim(\mathcal{C})+L)}. \tag{13}$$

Using these results we are now able to show that all possible $v^\top$ are indeed orthogonal to $\mathcal{C}\times\mathbb{R}^{mL}$. From the construction of the vectors w_i and equation (12) it follows that the final m entries of r, given by $\alpha_{\dim(\mathcal{C})}\eta_L$, must all be equal to zero as a consequence of equation (13), in other words $\eta_L = 0_m$. The sum (12) yields the preceding m entries of r as $\alpha_{\dim(\mathcal{C})}\eta_{L-1} + \alpha_{\dim(\mathcal{C})-1}\eta_L = \alpha_{\dim(\mathcal{C})}\eta_{L-1}$ which must again be equal to zero due to (13) giving $\eta_{L-1} = 0$. Repeating this procedure for all the $m(\dim(\mathcal{C}) + L)$ last elements of r we can show that

$$\eta_i = 0_m \tag{14}$$

for $i = 1, 2, ..., L$. Next, we move on to the first $m\dim(\mathcal{C})$ entries of r, which are now given by

$$\begin{bmatrix}\sum_{i=1}^{\dim(\mathcal{C})}\alpha_i\xi^\top\hat{A}^{i-1}B & \sum_{i=2}^{\dim(\mathcal{C})}\alpha_i\xi^\top\hat{A}^{i-2}B & \cdots & \alpha_{\dim(\mathcal{C})}\xi^\top B\end{bmatrix}$$

since all η_i-terms have been eliminated. In similar fashion to the technique used above, we can here observe from the last m elements that $\xi^\top B = 0_m^\top$ must be true in order for (13) to hold. This in turn means for the preceding m terms that

$$\sum_{i=\dim(\mathcal{C})-1}^{\dim(\mathcal{C})}\alpha_i\xi^\top\hat{A}^{i-1}B = \alpha_{\dim(\mathcal{C})-1}\xi^\top\hat{A}B = 0_m^\top,$$

giving $\xi^\top\hat{A}B = 0$. Again, repeating the procedure yields

$$\xi^\top\hat{A}^iB = 0_m^\top \tag{15}$$

for $i = 0, 1, ..., \dim(\mathcal{C}) - 1$. The equations (14) and (15) combined now guarantee that any vector v in the left kernel of the right-hand matrix in (7) is orthogonal to $\mathcal{C} \times \mathbb{R}^{mL}$, thus proving (7), and thereby the first statement of the theorem.

Finally, we seek to prove the second statement of the theorem, which asserts the existence of a vector g satisfying (4) for any trajectory of (1) originating in $\mathcal{C} + \overline{\mathcal{O}}$. Again, the proof is based on that in [7]. First, note that since the trajectory is a solution to the system (1) we know

$$\begin{bmatrix} \bar{u}_{[0,L-1]} \\ \bar{y}_{[0,L-1]} \end{bmatrix} = \begin{bmatrix} 0 & I \\ \mathbf{O}_L & T_L \end{bmatrix} \begin{bmatrix} x_0 \\ u_{[0,L-1]} \end{bmatrix}, \qquad (16)$$

where

$$T_L = \begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ CAB & CB & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{L-2}B & CA^{L-3}B & \cdots & D \end{bmatrix}$$

represents the impact of the control signal on the output. The matrix $\mathbf{O}_L$ is an observability matrix given by

$$\mathbf{O}_L = \begin{bmatrix} C^\top & (AC)^\top & \cdots & (A^{L-1}C)^\top \end{bmatrix}^\top, \qquad (17)$$

which corresponds to the influence of the initial condition on the output. Note that the definition is similar to (3) but its size depends instead on the length L. Here, if $x_0 \in \overline{\mathcal{O}}$ this directly implies that $x_0 \in \ker(\mathbf{O}_L)$ for $L \leq n$. For larger L the Cayley-Hamilton theorem can be used to show that $\text{im}(\mathbf{O}_L) = \mathcal{O}$.

Next, we can split the initial condition in two parts $x_0 = x_{\mathcal{C}} + x_{\overline{\mathcal{O}}}$, where the latter is the part lies in the unobservable subspace $\overline{\mathcal{O}}$ and the former is the remainder, which as a consequence lies in $\mathcal{C}$. By the reasoning above, then, we have that

$$\mathbf{O}_L(x_{\mathcal{C}} + x_{\overline{\mathcal{O}}}) = \mathbf{O}_L x_{\mathcal{C}}. \qquad (18)$$

The vector on the right-hand side of (18) can be parameterized according to the first statement of the theorem, since it lies in $\mathcal{C} \times \mathbb{R}^{mL}$. Finally, we combine this with (16) and (18) to get

$$\begin{bmatrix} \bar{u}_{[0,L-1]} \\ \bar{y}_{[0,L-1]} \end{bmatrix} = \begin{bmatrix} 0 & I \\ \mathcal{O}_L & T_L \end{bmatrix} \begin{bmatrix} \mathcal{H}_1(x_{[0,T-1]}) \\ \mathcal{H}_L(u_{[0,T-1]}) \end{bmatrix} g$$

which in turn simplifies to

$$\begin{bmatrix} \bar{u}_{[0,L-1]} \\ \bar{y}_{[0,L-1]} \end{bmatrix} = \begin{bmatrix} \mathcal{H}_L(u_{[0,T-1]}) \\ \mathcal{H}_L(y_{[0,T-1]}) \end{bmatrix} g,$$

thus proving the statement. ∎

IV. NUMERICAL EXAMPLE

In order to demonstrate the validity of the relaxed conditions for uncontrollable systems stated in Theorem 3.1 we make a comparison between controllable and uncontrollable systems with similar dynamics in the following example.

Example 4.1: Let a dynamical system on the form (1) be given with matrices

$$A = \begin{bmatrix} -0.2 & -0.2 & -0.2 & -0.2 & -0.2 & 0 \\ -0.2 & -0.2 & -0.2 & -0.2 & -0.2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0.1 & 0 & -0.2 & -0.1 & -0.1 & 0.1 \\ 0.1 & 0 & -0.2 & -0.1 & -0.1 & 0.1 \\ 0 & 0 & -0.1 & 0 & -0.3 & 0 \end{bmatrix}, \qquad (19)$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^\top, \quad C = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

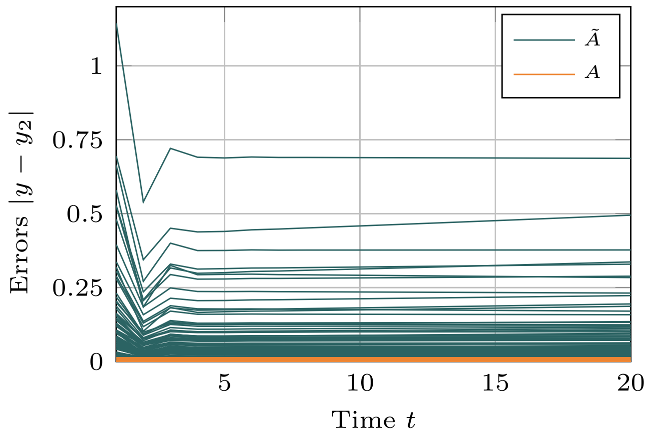
It is evident from equation (19) that the system loses three degrees of controllability; two due to the fact that $x_1 = x_2$ and $x_4 = x_5$ regardless of input signal u and one due to x_3 being independent of the other states and input.

In order to predict the output corresponding to a given input without knowing the matrices (19), we apply the data-driven simulation algorithm presented in [12, Section 4.2]. The algorithm uses a recorded input-output trajectory (u_d, y_d) to simulate the output corresponding to a given input and initial trajectory. Here, the output of interest is chosen to have a length of $L = 20$ time steps. To guarantee a unique and correct solution the method requires the recorded trajectory to be persistently exciting of order $T_{\text{ini}} + n + L$ where T_{ini} is the length of an initial trajectory. In the case of an observable system, which is the case for (19), we have the requirement that $T_{\text{ini}} = n$. Taken together, this results in a required order of presistency of excitation $T_{\text{ini}} + n + L = 32$. This would in turn require the recorded trajectory u_d to be *at least* 63 samples long according to Definition 4.

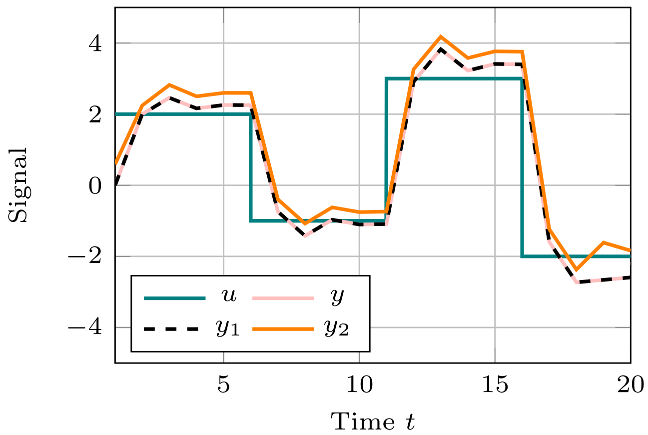
The condition on the degree of persistency of excitation originates from the condition in the original formulation of the fundamental lemma in [2]. As such, our Theorem 3.1 states that the term $n = 6$ can be reduced to $\dim(\mathcal{C}) = 3$ for the system (19) according to Theorem 3.1. Additionally, the length T_{ini} of the initial trajectory is bounded below by the order n of the system. However, the dynamics of the system given by (19) takes place in a subspace of dimension $\dim(\mathcal{C}) = 3$ under Assumption 1 and as a consequence, this condition can also be reduced. In total, this results in a requirement for the input signal to be persistently exciting of order 26, corresponding to a signal u_d that is at least 51 samples long.

Next, we consider another system similar to the one given by (19) but with dynamics governed by $\tilde{A} = A + E$ where E is a random, unstructured perturbation in all elements of A. This perturbed matrix $\tilde{A}$ will then with high probability be fully controllable. As such, Theorem 3.1 no longer applies and we require the full order of persistency of excitation $T_{\text{ini}} + n + L = 32$ to guarantee that the method works.

To show that the reduced data demand predicted for the uncontrollable system is correct, the two systems described above are simulated 100 times using the data-driven method from [12] with the same amount of data, $u_d \in \mathbb{R}^{51}$ (giving $(u_d, y_d) \in \mathbb{R}^{102}$), randomized for each individual trial. Figure 1 displays the resulting errors when the simulation is compared to a state-space simulation with the correct matrices given by (19) and known perturbation E applied to the second system, which is randomly sampled in each



(a) Error at each time step for 100 simulations of the uncontrollable system (orange) and the perturbed, and thus controllable, system (teal).



(b) The input signal u, actual output y and simulated output for the uncontrollable system y_1 and for the controllable (perturbed) system y_2.

Fig. 1: Data-based trajectory simulation relying on the condition on persistency of excitation. The uncontrollable system is correctly simulated up to very minor numerical errors. The controllable (perturbed) system displays significant errors varying in size between trials, see (a). In (b) the trajectory of the controllable system is shown for one trial. The simulation uses random input-output trajectories $(u_d, y_d) \in \mathbb{R}^{102}$ of each respective system as data.

trial. Both systems are initialized in the state $x = 0$, thus fulfilling Assumption 1 and the control signal during the initial trajectory is set to $u_{\text{ini}} = 0_{T_{\text{ini}}}$. As expected from the result in Theorem 3.1, the method is able to completely characterize the dynamics of the uncontrollable system using the reduced amount of data, giving a zero error. The perturbed system, however, is not confined to a lower-dimensional subspace ($\text{im}(\mathbf{C}) = \mathbb{R}^n$ since it is controllable). As a consequence, it displays errors in simulation with the data $(u_d, y_d)) \in \mathbb{R}^{102}$, which is not sufficient to excite all modes of the system.

V. Discussion

The results presented in Section III for the reduced demand on persistency of excitation are similar to those derived in [7]. Below we construct an example to show that the class of systems for which Theorem 3.1 give improved results in comparison to [7] is rich enough to be of interest. We then relate our results to the previous work on structural controllability in [11] and use the results presented therein to highlight a possible application of these results to realistic large-scale networks.

A. Comparison to previous results

To the best of our knowledge, [7] was the first to relax the requirements on persistency of excitation for certain systems lacking full controllability. The conditions stated therein implied that the requirement on the order of persistency of excitation could be reduced from the order n of the system to the degree of the minimal polynomial of the system matrix. In order to illustrate the existence and prevalence of systems that satisfy the conditions of Theorem 3.1 but do *not* fulfill the requirements stated in [7] we construct a simple example.

Example 5.1: Let the dynamics of a system on the form (1) be given by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

with arbitrary matrices C and D. We construct the controllability matrix according to Definition 1:

$$\mathbf{C} = \begin{bmatrix} 1 & a_{11} & \star \\ 0 & a_{21} & \star \\ 0 & a_{31} & \star \end{bmatrix}.$$

For the sake of this example we need not concern ourselves with the final column, which is here omitted for simplicity. It is evident that one condition that would lead $\mathcal{C}$ to lose rank is $a_{21} = a_{31} = 0$. Whenever this holds ($a_{21} = a_{31} = 0$) A has one eigenvalue located at a_{11} and two at the eigenvalues of the submatrix A'

$$A' = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}.$$

(note that the eigenvalues of A are given by the blocks on the diagonal). Therefore, the choice of a_{12} and a_{13} does not affect the location of the eigenvalues. The system will lose at least one degree of controllability regardless of the value of these matrix elements. However, the result in [7] only gives a reduced requirement on persistency of excitation if the degree of the minimal polynomial of A is less than n. This in turn is only possible if some eigenvalue has multiplicity greater than one.

The previous example demonstrates the existence of systems for which Theorem 3.1 requires a lower degree of persistent excitation than [7]. In addition, one would not in practice have access to the model of A used to find the minimal polynomial. Indeed, to avoid having to obtain this model is one of the main points of the data-driven approach. This is the second advantage of instead looking at the controllability of the system to determine the requirement for persistency of excitation. As will be shown below, methods exist for estimating the degree of controllability lost when actuating in only a few nodes of a network system. Furthermore, many real large-scale networks can be shown to significantly reduce the degree of controllability when this is the case.

B. Structural controllability of networks

Taking inspiration from [11] we can utilize the concept of structural controllability, first proposed in [9], to determine the applicability of Theorem 3.1 in sparsely actuated networks. Specifically, the authors of [11] demonstrate a computationally tractable method of empirically finding the number of actuator (or "driver") nodes N_D required to achieve full controllability of a system without knowing the weights of edges in the network. Actuating in only one of these nodes then necessarily means losing at least $N_D - 1$ degrees of controllability. Thus, assuming one is interested in the input-output behavior of a system from one node to some network-level output, it reduces the degree of persistent excitation required by Theorem 3.1 by the same number ($\dim(\mathcal{C}) \leq N_D - 1$). In networks with a high fraction $\frac{N_D}{n}$ of required actuators to total number of nodes this could entail a large reduction of the required amount of data compared to previous results. The study demonstrates the natural occurrence of several such systems, including a food web in Little Rock ($\frac{N_D}{n} = 0.541$), a power grid in Texas ($\frac{N_D}{n} = 0.325$) and a US corporate ownership framework ($\frac{N_D}{n} = 0.820$) which all require actuation in a significant fraction of nodes to reach controllability.

One criticism that has been levelled against the binary framework of (structurally) controllable vs uncontrollable modes in response to [11] is given in [13]. As is well known to control theorists, a quantitative measure of how difficult the modes of a system are to excite is often more relevant. In the context of our work, this would motivate taking a quantitative approach to the results presented in Section III. For example, is it possible to exploit not only loss of controllability, but also *almost* uncontrollable modes, to reduce the amount of data required by data-driven methods and what are the implications for data-driven control?

VI. Conclusion and directions for future work

We have proven a new lower condition on persistency of excitation for data-driven control and identification of uncontrollable systems. This, in combination with existing empirical methods to determine structural controllability, is used to motivate reduced data requirements in identifying the effects of local inputs on the output of certain large-scale networks.

One interesting avenue for future work would be the impact of neglecting modes that are difficult to excite. Exploring the extent to which this is possible without full knowledge of the system dynamics, as well as characterizing the loss of performance due to such an approximation, might imply further reduction of the amount of data required by model-free methods.

References

[1] K. Åström and B. Wittenmark, *Adaptive Control*, ser. Dover Books on Electrical Engineering. Dover: Dover Publications, 2008.

[2] J. C. Willems, P. Rapisarda, I. Markovsky, and B. L. De Moor, "A note on persistency of excitation," *Systems & Control Letters*, vol. 54, no. 4, pp. 325–329, 2005.

[3] I. Markovsky and F. Dörfler, "Behavioral systems theory in data-driven analysis, signal processing, and control," *Annual Reviews in Control*, vol. 52, pp. 42–64, 2021.

[4] J. Berberich, J. Köhler, M. Muller, and F. Allgöwer, "Data-driven model predictive control with stability and robustness guarantees," *IEEE Transactions on Automatic Control*, vol. 66, no. 4, pp. 1702–1717, 2021.

[5] J. Coulson, J. Lygeros, and F. Dörfler, "Data-enabled predictive control: In the shallows of the DeePC," in *2019 18th European Control Conference (ECC)*, 2019, pp. 307–312.

[6] H. Waarde, C. Persis, K. Camlibel, and P. Tesi, "Willems' fundamental lemma for state-space systems and its extension to multiple datasets," *IEEE Control Systems Letters*, vol. 4, no. 3, pp. 602 – 607, 04 2020.

[7] Y. Yu, S. Talebi, H. J. van Waarde, U. Topcu, M. Mesbahi, and B. Açıkmeşe, "On controllability and persistency of excitation in data-driven control: Extensions of Willems' fundamental lemma," in *2021 60th IEEE Conference on Decision and Control (CDC)*, 2021, pp. 6485–6490.

[8] V. Mishra, I. Markovsky, and B. Grossmann, "Data-driven tests for controllability," *IEEE Control Systems Letters*, vol. 5, no. 2, pp. 517–522, 4 2021.

[9] C.-T. Lin, "Structural controllability," *IEEE Transactions on Automatic Control*, vol. 19, no. 3, pp. 201–208, 1974.

[10] K. Glover and L. Silverman, "Characterization of structural controllability," *IEEE Transactions on Automatic Control*, vol. 21, no. 4, pp. 534–537, 1976.

[11] Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabasi, "Controllability of complex networks," *Nature*, vol. 473, pp. 167–73, 05 2011.

[12] I. Markovsky and P. Rapisarda, "Data-driven simulation and control," *International Journal of Control*, vol. 81(12), pp. 1946–1959, 12 2008.

[13] N. J. Cowan, E. J. Chastain, D. A. Vilhena, J. S. Freudenberg, and C. T. Bergstrom, "Nodal dynamics, not degree distributions, determine the structural controllability of complex networks," *PLOS ONE*, vol. 7, no. 6, pp. 1–5, 06 2012.