On the Coherence of Large-Scale Networks with Distributed PI and PD Control

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Networked systems: *global* objectives, but *local* feedback

Are there limitations to network *performance*?
Problem setup: Linear, second-order consensus subject to distributed disturbances

- Consider connected graph with $N$ agents
- Each agent $i$ is double-integrator
  \[
  \dot{x}_i(t) = v_i(t) \\
  \dot{v}_i(t) = u_i(t) + w_i(t)
  \]
- Control objective: follow trajectory $\bar{x}_i(t) := \bar{v}t + \delta_i$
- Standard linear consensus / Proportional (P) control
  \[
  u_i = -\sum_{j \in \mathcal{N}_i} f_{ij}(x_i - x_j) - \sum_{j \in \mathcal{N}_i} g_{ij}(v_i - v_j) - f_0 x_i - g_0 v_i
  \]
  Relative feedback
  Absolute feedback
- Let each agent be subject to stochastic disturbance $w_i(t)$

$x_i, v_i$ deviation from state trajectory, $\delta_i$ setpoint, $f_{ij}, g_{ij}, f_0, g_0$ fixed, constant gains, $\mathcal{N}_i$ neighbor set
Example 1: Large-scale vehicle platoons

- Objective: follow trajectory \( \bar{x}_i(t) := \bar{v}t + \delta_i \)
  - common cruising speed \( \bar{v} \)
  - tight constant spacing \( \Delta \), so that \( \delta_i = \Delta i \)

- Example control law: look-ahead, look-behind control

\[
u_i = f^+(x_{i+1} - x_i) + f^-(x_{i-1} - x_i) + g^+(v_{i+1} - v_i) + g^-(v_{i-1} - v_i)
\]

\((f^+, f^-, g^+, g^- \text{ constant gains})\)

- With disturbances: objectives only achieved approximately

- What happens if the platoon grows?
Example 1 (contd.): Performance issues if control based on relative feedback

Formation is stable
Spacings $\leftrightarrow$ are well-regulated (no collisions!)
However - not a rigid formation, not coherent!
Fundamental limitation to local, static feedback (Bamieh et al., 2012)

Can dynamic feedback (PID control) help?
Example 2: Frequency control in power networks

- Objectives:
  - common, steady frequency $\bar{\omega}$ (60 Hz)
  - phase angles at equilibrium $\left(\theta_i - \theta_j\right) \sim P_i^*$

- Swing equation, or “droop control” (linearized)

\[
m_i \dot{\omega}_i = -d_i \omega_i - \sum_{j \in N_i} b_{ij} (\theta_i - \theta_j) + w_i
\]

(bij line susceptance, mi inertia, di damping)

- Transition to distributed generation affects power system dynamics
  - More disturbances, (many) more generators
Example 2 (contd.): Issues with scalability of standard droop controller

- Simulation of droop control on 10 vs 100 node network (tree graph)

- **Today**: Better scalability with distributed PI-control (dynamic feedback)
Problem setup: Performance is quantified through a measure of network coherence

- Consider each agent’s deviation from the network average
  \[ y_i^{\text{dev}} = x_i - \frac{1}{N} \sum_{j=1}^{N} x_j \]

- Characterizes rigidity, coherence

- Performance is measured as *variance* of performance output, normalized by \( N \)
  \[ V_N = \frac{1}{N} \mathbb{E}\{y^T(t)y(t)\} \]

- Interested in the *scaling* of the output variance with network size
Summary: We characterize scalability of distributed control laws

• **Model:** Second order consensus with performance output
  \[
  \begin{bmatrix}
    \dot{x} \\
    \dot{v}
  \end{bmatrix} = \begin{bmatrix}
    0 & I \\
    -\mathcal{L}_F - f_0 I & -\mathcal{L}_G - g_0 I
  \end{bmatrix} \begin{bmatrix}
    x \\
    v
  \end{bmatrix} + \begin{bmatrix}
    0 \\
    I
  \end{bmatrix} w
  \]
  \[
  y = \begin{bmatrix}
    I - \frac{1}{N}11^T & 0
  \end{bmatrix} \begin{bmatrix}
    x \\
    v
  \end{bmatrix}
  \]

  - Absolute feedback from \(x\) (\(v\)) if \(f_0\) (\(g_0\)) nonzero

• **Performance evaluation:**
  - Consider (asymptotic) scaling of variance
    \[V_N = \frac{1}{N} \mathbb{E}\{|y(t)^T y(t)|\}\]
  - Control law scales well only if \(V_N\) bounded in \(N\)

• **Objective:** Compare static vs. dynamic feedback

(\(\mathcal{L}_F, \mathcal{L}_G\) weighted graph Laplacians, assume \(\mathcal{L}_F = f\mathcal{L}, \mathcal{L}_G = g\mathcal{L}\) for some (weighted) \(\mathcal{L}\))
OUTLINE

Introduction and problem formulation

$H_2$

Evaluating input-output performance

Distributed PI and PD control

Conclusions and future work

On the Coherence of Large-Scale Networks with Distributed PI and PD Control

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Performance is evaluated through input-output $H_2$ norms

Consider general linear system under white noise input

$$\begin{align*}
H : & \quad \dot{x} = Ax + Bw \\
& \quad y = Cx \quad (1)
\end{align*}$$

Recall:
Need to evaluate $V_N = \frac{1}{N} \mathbb{E}\{y^T y\}$, with $y = (I_N - \frac{1}{N} 11^T)x$

Lemma:
The squared $H_2$ norm of (1) from input $w$ to output $y$ gives

$$||H||_2^2 = \lim_{t \to \infty} \mathbb{E}\{y^T(t)y(t)\},$$

That is, the steady state output variance.

Evaluating system performance amounts to evaluating $H_2$ norms!
Eigenvalues near zero cause bad performance

**Theorem**

\[ V_N = \frac{1}{N} \|H\|^2 = \frac{1}{2N} \sum_{n=1}^{N-1} \frac{1}{(f_0 + f\lambda_n)(g_0 + g\lambda_n)} \]

**Example (Ring graph, uniform weights):**

- Eigenvalues
  \[ \lambda_n = 2 \left( 1 - \cos \frac{2\pi n}{N} \right) \]

- As \( N \) grows: Arbitrarily many \( \lambda_n \) increasingly close to zero
- Sum blows up, unless \( f_0, g_0 \neq 0 \), i.e., absolute feedback

Precise scaling of \( V_N \) in \( N \) can be determined for regular graphs
P-control scales badly in sparse networks, unless absolute feedback available.

- **Recall:**
  \[ u_i = -\sum_{j \in \mathcal{N}_i} f_{ij}(x_i - x_j) - \sum_{j \in \mathcal{N}_i} g_{ij}(v_i - v_j) - f_0 x_i - g_0 v_i \]
  - **Relative feedback**
  - **Absolute feedback**

- Let network be \( d \)-dimensional lattice

\[ d = 1 \quad d = 2 \quad d = 3 \]

Asymptotic performance scalings with static feedback (see e.g. Bamieh et al., 2012)

*Up to a constant independent of gain parameter \( \beta \) and network size \( N \)*

<table>
<thead>
<tr>
<th>Relative ( x ), relative ( v )</th>
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</table>
| \( V_N \sim \frac{1}{\beta^2} \left\{ \begin{array}{ll} N^3 & d = 1 \\
             N & d = 2 \\
             N^{1/3} & d = 3 \\
             \log N & d = 4 \\
             1 & d \geq 5 \end{array} \right. \) | \( V_N \sim \frac{1}{\beta} \left\{ \begin{array}{ll} N & d = 1 \\
                                                   \log N & d = 2 \\
                                                   1 & d \geq 3 \end{array} \right. \) | \( V_N \sim \frac{1}{\beta} \) |
Various strategies proposed to deal with performance limitations

- Assign select leaders with absolute measurement (1st order consensus)
  - S. Patterson et al. “Leader selection for optimal network coherence,” CDC 2010
  - M. Pirani et al. “Coherence and convergence rate in networked dynamical systems,” CDC 2015

- Optimize gains, change symmetries

- Here: use distributed PID-control
  - D. Lombana and M. di Bernardo, “Distributed PID control for consensus of homogeneous and heterogeneous networks, TCNS 2016
Idea: use derivative or integral action to substitute unavailable measurement

**Derivative action**
Absolute $x$-measurement

- Derivative of $x$-measurement corresponds to $v$
  \[
  \frac{dx_i}{dt} = v_i(t)
  \]
- *Ideally:* same performance as with absolute feedback in $x, v$
- Ideal derivative action not possible to implement + sensitive to noise

**Integral action**
Absolute $v$-measurement

- Integral of $v$-measurement corresponds to $x$
  \[
  \int_0^t v_i(\tau)d\tau = x_i(t) - x_i(0)
  \]
- *Ideally:* same performance as with absolute feedback in $x, v$
- Ideal derivative action not possible to implement
- Decentralized integration does not give robustly stable system

Modifications of the control laws required to enable implementation
Filtered distributed PD-control (F-DPD)

- Control law: (Laplace domain!)

\[ U_i = -\sum_{j \in \mathcal{N}_i} f_{ij}(X_i - X_j) - \sum_{j \in \mathcal{N}_i} g_{ij}(V_i - V_j) - f_0 X_i - \frac{s}{\tau s + 1} K_D X_i \]

- Low-pass filter prevents too large variations in control signal

Theorem

\[ V_{N}^{F-\text{DPD}} = \frac{1}{2N} \sum_{n=2}^{N} \left( f_0 + f \lambda_n \right) \left( g \lambda_n + \frac{K_D (\tau g \lambda_n + 1)}{\tau^2 (f_0 + f \lambda_n) + \tau g \lambda_n + 1} \right) \]

For any positive \( K_D \) and \( \tau \), \( V_{N}^{F-\text{DPD}} \) is uniformly bounded in \( N \) for any network:

\[ 0 < V_{N}^{F-\text{DPD}} < \frac{\tau^2 f_0 + 1}{2 f_0 K_D} \]

- Higher order filters give same result
- Theoretical performance best if filter constant \( \tau = 0 \)
Distributed averaging PI-control (DAPI) 1(2)

- Control law:
  \[
  u_i = -\sum_{j\in\mathcal{N}_i} f_{ij}(x_i - x_j) - \sum_{j\in\mathcal{N}_i} g_{ij}(v_i - v_j) - g_0 v_i - K_I z_i
  \]
  \[
  \dot{z}_i = -v_i - \sum_{j\in\mathcal{N}_i} c_{ij}(z_i - z_j)
  \]

- Distributed averaging filter prevents de-stabilizing drift by aligning integral state

- Proposed in power system context (secondary frequency control)

**Theorem**

Assume uniform ratios \(c_{ij}/f_{ij}\), so \(\mathcal{L}_c = c\mathcal{L}\), then

\[
V_N^{\text{DAPI}} = \frac{1}{2N} \sum_{n=2}^{N} \frac{1}{fg\lambda_n^2 + \frac{K_I f(g_0 + \lambda_n(c+g)) + g_0 f\lambda_n(c^2\lambda_n + f + cg_0)}{f + cg_0 + c\lambda_n(c+g)}}.
\]

For any positive and finite \(K_I\) and \(c\), \(V_N^{\text{DAPI}}\) is uniformly bounded in \(N\):

\[
0 < V_N^{\text{DAPI}} < \frac{f + cg_0}{2K_I fg_0}.
\]
Distributed averaging PI-control (DAPI) \(2(2)\)

- Design of distributed averaging filter affects performance
  - \(c \to 0 \Rightarrow z_i \approx \int_0^t v_i(\tau) d\tau = x_i\)
  - \(c \to \infty \Rightarrow\) same perf. as w/o PI control
  - In some cases, optimal \(c^* > 0\)

**Corollary (optimal distr. averaging)**

The optimal gain \(c^* > 0\) if

\[
f > \frac{1}{\lambda_n} (g\lambda_n + g_0)^2,
\]

for all \(n = 2, \ldots, N\).

- For insights to optimal topology, see X. Wu et al. (ACC, 2016), D. Deka et al. (ACC, 2017)
**Summary:** PI and PD control can relax performance limitations

<table>
<thead>
<tr>
<th>Standard consensus</th>
<th>Relative $x$, relative $v$</th>
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<td>F-DPD, DAPI</td>
<td>N/A</td>
<td>$V_N \sim \frac{1}{\beta}$</td>
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($\beta$ parameter reflecting control effort, $d$ lattice dimension)
Example 1: F-DPD in vehicular formation

- Assume no speedometer, but position is known
- Compare standard protocol to F-DPD

Subset of 100 vehicle platoon, simulated under white noise disturbance
Example 2: DAPI in frequency control

- In power networks, frequency $\omega_i$ can be measured, but measurement of phase $\theta_i$ requires phasor measurement unit (PMU)
- DAPI improves performance and scalability, +eliminates stationary error

Simulation of synchronization transient in radial network with $N=10$ and $N=100$ nodes
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Ongoing and future work

- Can scalings at all be improved without absolute measurements?
- Issues with measurement noise and bias
- Further applications in power networks:
  - Scalability of frequency control
  - Use of PMUs
Thank you!

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Funding support from: